Accelerated Life Models for Failure-Degradation Data and Thier Application in Reliability and Maintenance

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La Rochelle, 5 Octobre, 2011

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1. Introduction and Notations.

Failures of **highly reliable units are rare**.

One way of obtaining a **complementary reliability information** is to do **accelerated life testing** (**ALT**), i.e. **to use higher level of experimental factors**, hence **to obtain failures quickly**.

Another way of obtaining **complementary reliability information** is **to measure** some parameters which characterize the **aging, wear or degradation of the product in time**.

Statistical inference from **ALT is possible** if **failure time regression models relating failure time distribution with external explanatory variables** (covariates, stresses) influencing the reliability are **well chosen**.

We suppose that the following data are available for reliability characteristics estimation: **failure times** (possibly **censored**), **explanatory variables** (covariates, stresses) and the values of some **observable quantity characterizing the degradation** of units. The failure rate of units may **depend** on **covariates, degradation level and time**. For example, many covariates influence the wear rate of tires: state and type of road covering, weight of the load, weather conditions (temperature, humidity), pressure inside tires, type of a vehicle, steep turns, etc. Via the wear and directly the **covariates** may **influence** the **intensity of traumatic failures**.

We consider a possibility to study the reliability data when the **degradation is modelled by random process**. We call a failure **non-traumatic** when the **degradation attains a critical level** z_0. Other failures are called **traumatic**. **Traumatic failures** may be of **different modes**: related with production defects, caused by mechanical damages or by fatigue of components. We shall consider the simplest case when only the failure of **one mode** is observed.

In this talk we consider estimation of **degradation and failure process characteristics using degradation and failure time data with covariates** supposing **differently in the generalization of the Cox model** that **the components of hazard rate related with observable degradation is unknown function of degradation and may depend on covariates**.

Wulfsohn and Tsiatis (1997) proposed the so-called **joint model** for **survival data** and **longitudinal data** **measured with error**, according to which the **conditional hazard rate** $\lambda_T(t|A=a)$ of the **time to traumatic failure** T has the form

$$\lambda_T(t|A=a) = \lambda_0(t)\exp^{\beta(a_1+a_2t)}, \quad t>0,$$

where $Z(t) = A_1 + A_2 t$ is the **observed degradation process** considered as a **random covariate**, $A = (A_1, A_2)^T$ follows **bivariate normal distribution**, $\lambda_0(t)$ is the **unknown baseline hazard function**, as in the **Cox model**.
In 1997/1998 Meeker and Escobar (1998) proposed to use the so called **path models** to have simple degradation processes:

$$Z(t) = \frac{t}{A}, \quad Z(t) = \left(\frac{t}{A_1}\right)^{A_2}, \quad Z(t) = g(t, A),$$

g is an **increasing** function in t.

Bagdonavicius and Nikulin (2001, 2003) proposed the next model in terms of **conditional survival function** of the **time to traumatic failure** T given the **real degradation process**:

$$S_T(t|A) = \mathbf{P}\{T > t | g(s, A), 0 \le s \le t\} =$$
$$\exp\left\{-\int_0^t \lambda_0(s, \theta)\lambda(g(s, A))ds\right\},$$

where $\lambda(t)$ is the **unknown intensity function**, $\lambda_0(s, \theta)$ being from **parametric family of hazard functions**. The distribution of A is **not specified**. This model, where the degradation is presented in explicit form as a random covariate, states that the **conditional hazard rate** $\lambda_T(t, A)$ at the moment t given the **real degradation** $g(s, A), 0 \le s \le t$, has the **multiplicative form** as in the **famous Cox model**:

$$\lambda_T(t|A = a) = \lambda_0(t, \theta)\lambda(g(t, A)), \quad t > 0.$$

According to this model $B\&N$ suppose that we **observe** (**measure**) the degradation process with a **noise**:

$$Z(t) = Z_r(t)U(t), \quad t \geq 0, \quad \sigma > 0,$$

where $Z_r(t), t \geq 0$, is the process of the **real degradation**, often it is supposed that $Z_r(t) = g(t, A)$, and

$$\ln U(t) = \sigma W(c(t)), \quad c(0) = 0,$$

where W is the **standard Wiener process** **independent** on A, and $c : [0, \infty) \to [0, \infty)$ is an **increasing continuous** function.

We consider the so-called **dynamic regression models** with **time depending covariates** which are **well adapted** to study the phenomena of **longevity, aging, fatigue and degradation of complex systems**, and hence appropriate to be used in the organization of the efficient **statistical process of quality control** in **dynamic environments**.

The **ALT** method is described very well in statistical literature, see, Singpurwalla (1971), Viertl (1988), Meeker and Escobar (1997), Bagdonavicius and Nikulin (2002), Lawless (2003), Nelson (2004) Vonta, Nikulin, Limnios, Huber-Carol (Eds.) (2008), Huber, Limnios, Mesbah, Nikulin (Eds.)(2008), Nikulin, Limnios, Balakrishnaan, Kahle, Huber (2010), Rykov, Balakrishnan, Nikulin (2010).

Denote by T the random **time-to-failure** of a unit (or system). We say also that T is the time of **hard** or **traumatic failure**. Let S be the **survival function** and λ be the **hazard rate**:

$$S(t) = \mathbf{P}\{T > t\}, \quad \lambda(t) = \lim_{h \to 0} \frac{1}{h} \mathbf{P}\{t \le T < t+h | T \ge t\} = -\frac{d[\ln S(t)]}{dt}, \tag{1}$$

from where it follows that $S(\cdot)$ can be written as

$$S(t) = e^{-\Lambda(t)}, \quad \text{where} \quad \Lambda(t) = \int_0^t \lambda(s) ds$$

is the **cumulative hazard function**. $F(\cdot) = 1 - S(\cdot)$ is the **cumulative distribution function** of T. One can say also that $\Lambda(t), t \ge 0$, is the **deterministic process** of **natural degradation**.

In Survival Analysis and Reliability the models are often formulated in terms of **cumulative hazard** and **hazard rate functions**. The most common **shapes of hazard rates** are **monotone, $\cup$- shaped or $\cap$- shaped**, see, for example, Bagdonavičius, Clerjaud and Nikulin (2007).

If the form of the survival or other equivalent function is supposed to be known and this function depends only on finite dimensional parameter the corresponding model is **parametric**. Classes of parametric models used in reliability are given in Meeker and Escobar (1998), Bagdonavičius and Nikulin (2002), Lawless (2003), Nikulin, Limnios, Balakrishnan, Kahle and Huber (2010).

Suppose that the **explanatory variable** is a deterministic time function

$$x = x(\cdot) = (x_1(\cdot), \ldots, x_m(\cdot))^T : [0, \infty[\to B \in R^m,$$

which is a **vector of covariates** itself or a realization of a stochastic process $X(\cdot)$, which is called also the **covariate process**, $X(\cdot) = (X_1(\cdot), \ldots, X_m(\cdot))^T$. Note that the covarite process could play the role of the degradation process also. We denote by E the set of **all possible (admissible) covariates** and by E_1 – the set of all **constant over time** covariates, $E_1 \subset E$.

We want to understand how we may to construct the survival models when the distributions of reliability data, in particular of the degradation data, depend on covariates.

We do not discuss here the questions of choice of X_i and m, but they are very important for the organization (design) of the experiments and for statistical inference. The covariates can be interpreted as the **control** (see Ceci and Mazliak, (2004)), since we may consider models of aging in terms of differential equations and so to use all theory and techniques from the optimal control theory. We may say that we consider statistical modeling with **dynamic design** or in **dynamic environments**.

Let E_2, $E_2 \subset E$, be a set of **step-stresses** of the form

$$x(t) = x_1 \mathbf{1}_{\{0 \le t < t_1\}} + x_2 \mathbf{1}_{\{t_1 \le t\}}, \quad x_1, x_2 \in E_1. \tag{2}$$

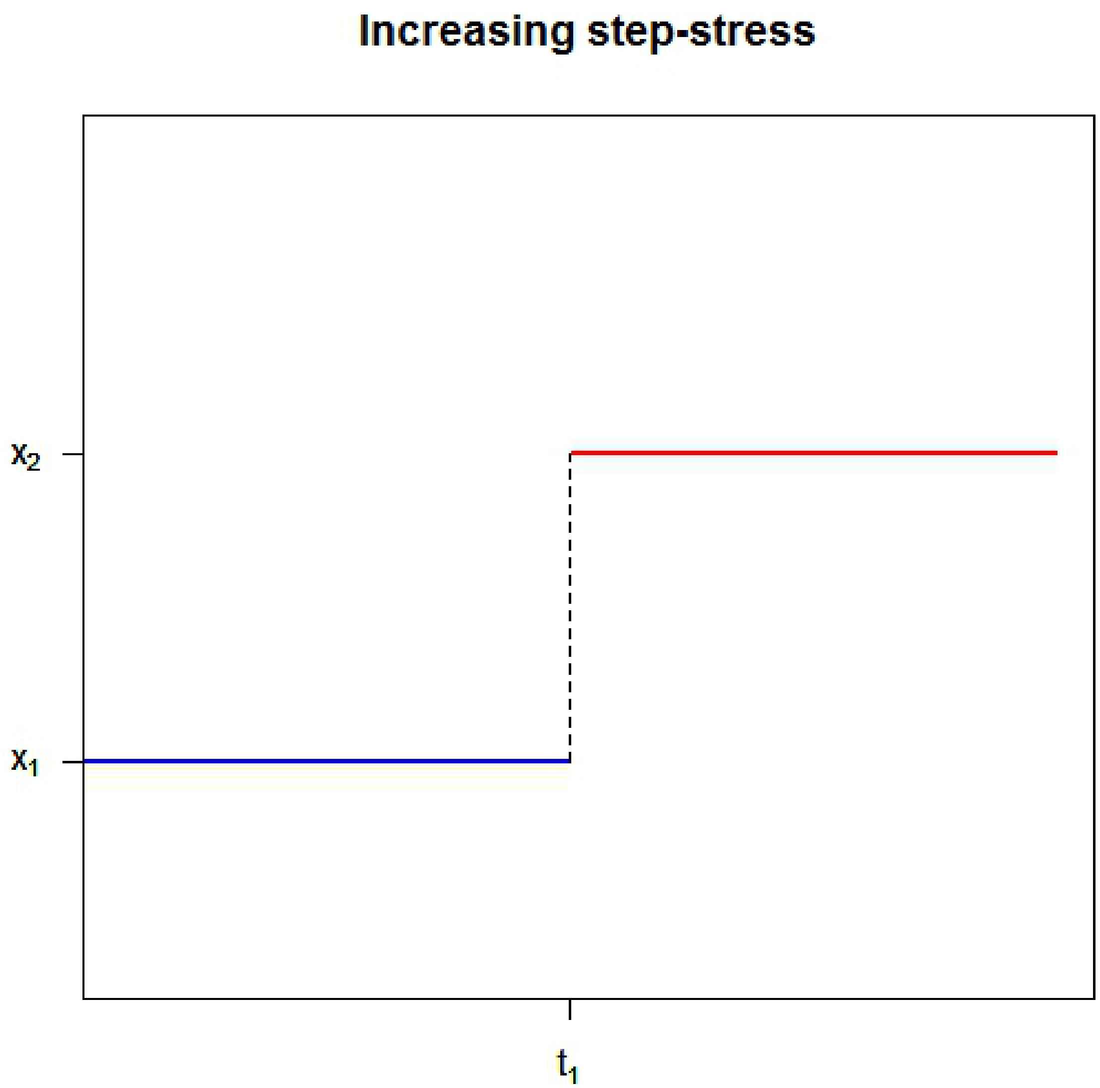

Fig.1. Increasing step-stress for the warm stand-by unit.

In accordance with (1) the survival, the hazard rate, the cumulative hazard and the distribution functions given $x \in E$ are :

$$S(t|x) = \mathbf{P}\left(T > t|x(s); 0 \leq s \leq t\right), \ \lambda(t|x) = -\frac{S'(t|x)}{S(t|x)},$$

$$\Lambda(t|x) = -\ln\left[S(t|x)\right], \ F(t|x) = \mathbf{P}\left(T \leq t|x(s); 0 \leq s \leq t\right) = 1 - S(t|x),$$

from where one can see their dependence on the **life-history** up to time t. We denote $x_0(\cdot) \in E$ a fixed **"normal" stress** which is used in the **"normal" or "standard"** conditions. Often x_0 is a **constant over time** covariate. Let x and y be two **admissible** stresses: $x, y \in E$. We say that a stress y is **accelerated** with respect to x, if

$$S(t|x) \geq S(t|y), \quad \forall\, t \geq 0, \quad S(\cdot|x), S(\cdot|y) \in \{S(\cdot|z), z \in E\}.$$

On any family E of **admissible stresses**, we may consider a class $\{S(\cdot|x), x \in E\}$ of survival functions which could be very rich. We say that the time $f(t|x)$ under the **normal stress** x_0 is **equivalent** to the time t under the stress x if the **probability that a unit used under the stress x would survive till the moment t** is **equal** to the **probability that a unit used under the stress x_0 would survive till the moment $f(t|x)$**:

$$S(t|x) = \mathbf{P}\{T > t|x(s); 0 \le s \le t\} =$$

$$\mathbf{P}\{T > f(t|x)|x_0(s); 0 \le s \le f(t|x)\} = S(f(t|x)|x_0).$$

More shortly, for any $x \in E$ we can write

$$S(t|x) = S(f(t|x)|x_0).$$

It implies that

$$f(t|x) = S^{-1}\left[S(t|x)|x_0\right], \quad x \in E. \tag{3}$$

It is natural to assume that the **distribution of the degradation process**

$$Z_x(t) = Z(t|x), \quad t > 0,$$

observed under stress $x(\cdot), x(\dot{}) \in E$, at the moment t is the same as the distribution of degradation process, observed under stress x_0, at the moment $f(t|x)$:

$$Z(t|x) = Z(f(t|x)|x_0), \quad t > 0.$$

This model can be practically used if a concrete form of the functional $f(t|\cdot)$ is assumed, i.e. an accelerated life model relating failure time to stress is given.

2. Accelerated Life and failure Time Regression models

Failure time regression models relating the **lifetime distribution** to possibly time dependent external explanatory variables are considered in this section. Failure time **regression models relating failure time distribution** not only with external but also with internal explanatory variables will be discussed in the next section. Such models are used now not only in reliability but also in **demography, dynamics of populations, gerontology, biology, survival analysis, genetics, radiobiology, biophysics**, everywhere people study the problems of **longevity, aging and degradation** using the stochastic modeling.

In reliability, **ALT** in particular, the choice of **a good regression model** often is **more important** than in **survival analysis**. For example, in **ALT** units are tested under **accelerated stresses** which shorten the life. Using such experiments the life **under the usual stress** is estimated using **some regression model**. The values of the **usual stress** are often **not in the range** of the values of accelerated stresses, since the wide separation between experimental and usual stresses is possible, so if the **model is misspecified**, the estimators of survival under the usual stress **may be very bad**.

2.1. Sedyakin's model

The **physical principle in reliability**, proposed by N. Sedyakin in (1966), gives an interesting way to prolong any class of survival functions $\{S(\cdot|x), x \in E_1\}$ indexed by constant in time stresses to a class of survival functions indexed by **step-stresses**, for example from E_2 , given by (2).

According to Sedyakin we may consider the next model on E_2: if two moments t_1 and t_1^* are **equivalent**, i.e. the probabilities of survival until these moments under stresses x_1 and x_2, respectively, are **equal**, i.e.

$$S(t_1|x_1) = S(t_1^*|x_2),$$

then

$$\lambda(t_1 + s|x) = \lambda(t_1^* + s|x_2), \quad \forall s \geq 0. \tag{4}$$

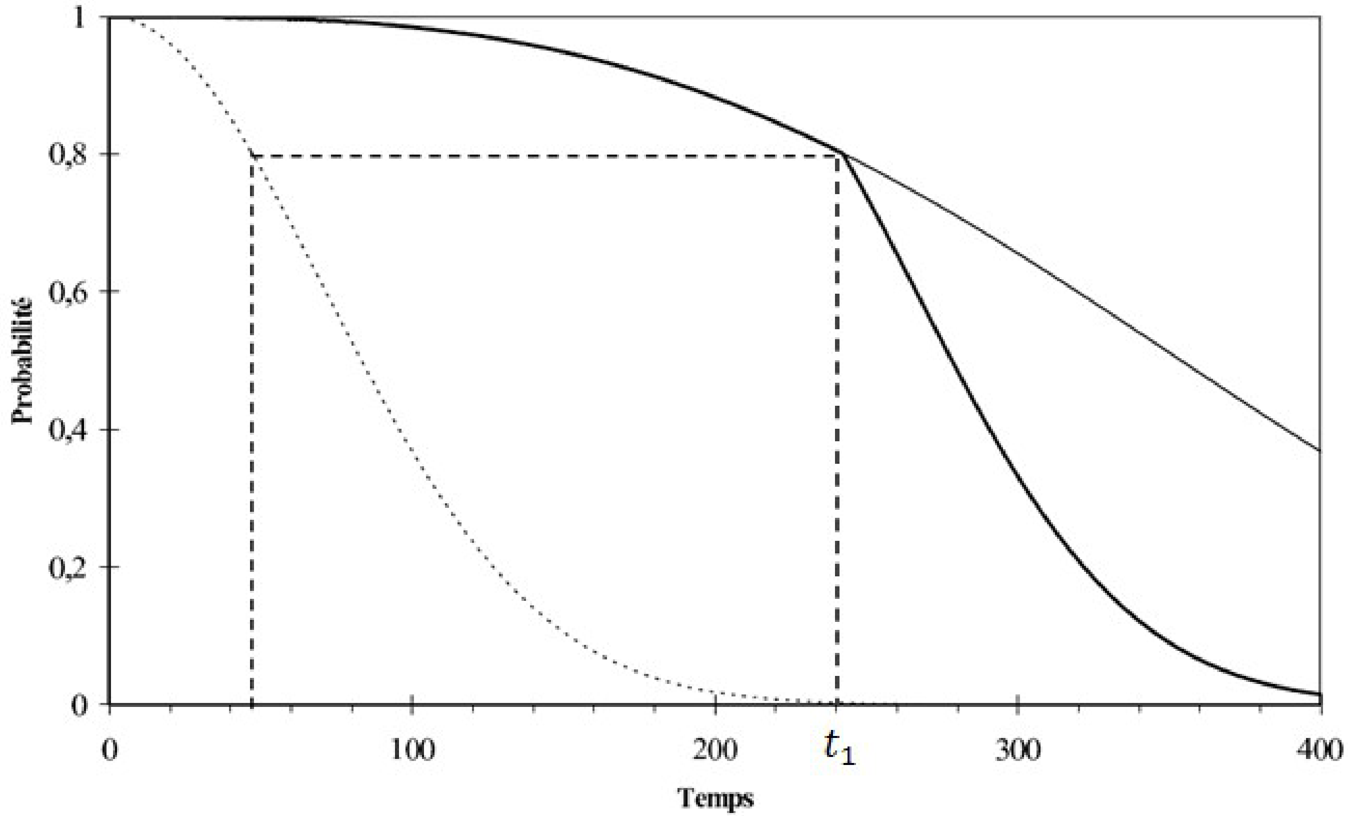

Fig.2. Sedyakin model on E_2 for $x_1 < x_2$.

The meaning of this **rule of time-shift** for these **step-stresses** on E_2 one can see also in terms of the survival functions:

$$S(t|x) = \begin{cases} S(t|x_1), & 0 \le t < t_1, \\ S(t - t_1 + t_1^*|x_2), & t \ge t_1. \end{cases} \tag{5}$$

The model given by (4) (or (5)) is called the **Sedyakin's model** on E_2.

The general Sedyakin (GS) model (Bagdonavičius and Nikulin (1995)) generalizes this idea, by supposing that the hazard rate at any moment t depends on the stress at this moment and on the probability of survival until this moment :

$$\lambda(t|x) = g\left(x(t), S(t|x)\right), \quad x \in E.$$

Bagdonavičius and Nikulin (2002) give **goodness-of-fit test** for the **GS** model.

The Sedyakin's model is too wide for **ALT** and failure time regression data analysis. It only states that units which did not fail until equivalent moments under different stresses have the same risk to fail after these moments under identical stresses. Nevertheless, this model **is useful for construction** of narrower models and for analysis of **redundant systems**.

2.2. Accelerated Failure Time Model

The accelerated failure time (**AFT**) model is more adapted for failure time regression analysis (see Bagdonavičius (1978), Viertl (1988), Meeker and Escobar (1998), Lawless (2003), Nelson (2004)). In **ALT** it is **the most used model**.

We say that **AFT** model holds on E if there exists a positive continuous function $r : E \to \mathbf{R}^1$ such that for any $x \in E$ the survival and the cumulative hazard functions under a covariate realization $x \in E$ are given by formulas:

$$S(t|x) = G\left(\int_0^t r\left[x(s)\right] ds\right) \quad \text{and} \quad \Lambda(t|x) = \Lambda_0\left(\int_0^t r\left[x(s)\right] ds\right), \tag{6}$$

respectively, where $G(t) = S(t|x_0)$, $\Lambda_0(t) = -\ln G(t)$, x_0 is a **given (usual) stress**, $x_0 \in E$. The function r **changes locally the time scale**. From the definition of $f(t|x)$ (cf. (3)) it follows that for the **AFT** model on E

$$f(t|x) = \int_0^t r\left[x(s)\right] ds, \quad \text{hence} \quad \frac{\partial f(t|x)}{\partial t} = r(x(t)) \tag{7}$$

at the continuity points of $r[x(\cdot)]$. Note that the model can be considered as **parametric** (r and G belong to parametric families), **semiparametric** (one of these functions is unknown, other belongs to a parametric family) or **nonparametric** (both are unknown).

For example, if we consider the **Cox's parametrization** of $r(x(s)) = e^{\beta^T x(s)}$ then we shall have the next **AFT** model

$$S(t|x) = G\left(\int_0^t e^{\beta^T x(s)} ds\right), x(\cdot) \in E,$$

where $G(t) = S(t|x_0)$, x_0 is a **given (usual) stress**, $x_0 \in E$, and β is the regression parameter having the **same dimension as** x. From the definition of $f(t|x)$ (cf. (3)) it follows that under such parametrization of r the **time transformation in dependence on** s is given by formula:

$$f(t|x) = f(t, x, \beta) = \int_0^t e^{\beta^T x(s)} ds, \quad x(\cdot) \in E_1.$$

In particular, if $x(\cdot) = x$, $x \in E_1$, then

$$f(t|x) = f(t, x, \beta) = e^{\beta^T x} t, \quad x \in E_1.$$

The **AFT** model can be also given by the next formula:

$$\lambda(t|x) = r\left(x(t)\right) q\left(\Lambda(t|x)\right), \quad x \in E.$$

This equality shows that, differently from the famous **Cox model**, the hazard rate $\lambda(t|x)$ **is proportional** not only to some function of the stress at the moment t but also to a function of the cumulative hazard $\Lambda(t|x)$ at the moment t. It means that the hazard rate at the moment t **depends not only on the stress applied at this moment** but also on the stress applied in **the past**, i.e. in the interval $[0, t)$.

In **parametric modeling** a **baseline survival function** G is taken from some class of parametric distributions such as Weibull, lognormal, log-logistic, Inverse Gaissian, Birnbaum-Saunders, etc., and the function r is taken in the form $r(x) = e^{\beta^T \varphi(x)}$, where $\varphi(x)$ is a specified possibly multidimensional function of x (see, Zacks (1992), Nelson (2004)). Versatile model is obtained when G belongs to the **power generalized Weibull** (**PGW**) family of distributions (see, Bagdonavičius and Nikulin (2002)). In terms of the survival functions the **PGW** family is given by the next formula:

$$S(t,\sigma,\nu,\gamma) = exp\left\{1 - \left[1 + \left(\frac{t}{\sigma}\right)^{\nu}\right]^{\frac{1}{\gamma}}\right\}, \quad t > 0, \gamma > 0, \nu > 0, \sigma > 0.$$

If $\gamma = 1$ we have the Weibull family of distributions. If $\gamma = 1$ and $\nu = 1$, we have the **exponential family** of distributions. This class of distributions has nice probability properties. All moments of this distribution are finite. For various values of the parameters the hazard rate can be **constant, monotone** (increasing or decreasing), **unimodal** or $\bigcap$-shaped, and **bathtub** or $\bigcup$-shaped.

Semiparametric estimation procedures for the **AFT** model are given in Lin and Ying (1995) and developed by many authors, **non-parametric estimation procedures** with special plans of experiments are given in Bagdonavičius and Nikulin (2000).

The **AFT** model is popular in reliability theory because of its **interpretability**, its nice mathematical properties and its consistency with some engineering and physical principles. Nevertheless, the assumption that the survival distributions under different covariate values **differ only in scale is rather restrictive**.

2.3. Application of Sedyakin's and AFT models for redundant systems modelling

Let us consider **redundant systems** with one **main unit** operating in **"hot"** conditions and $m-1$ **stand-by units** operating in **"warm"** conditions, i.e. under **lower stress** than the main one. We shall use notation $S(1, m-1)$ for such systems.

If the main unit fails then one **stand-by units** from those not yet failed **is commuted and operates** instead of the main one. We suppose that **commuting is momentary** and there are no repairs.

We suppose that switching from **"warm"** to **"hot"** conditions **does not do any damage** to units. Mathematical formulation of **"fluent switch on"** is based on the **Sedyakin's model**.

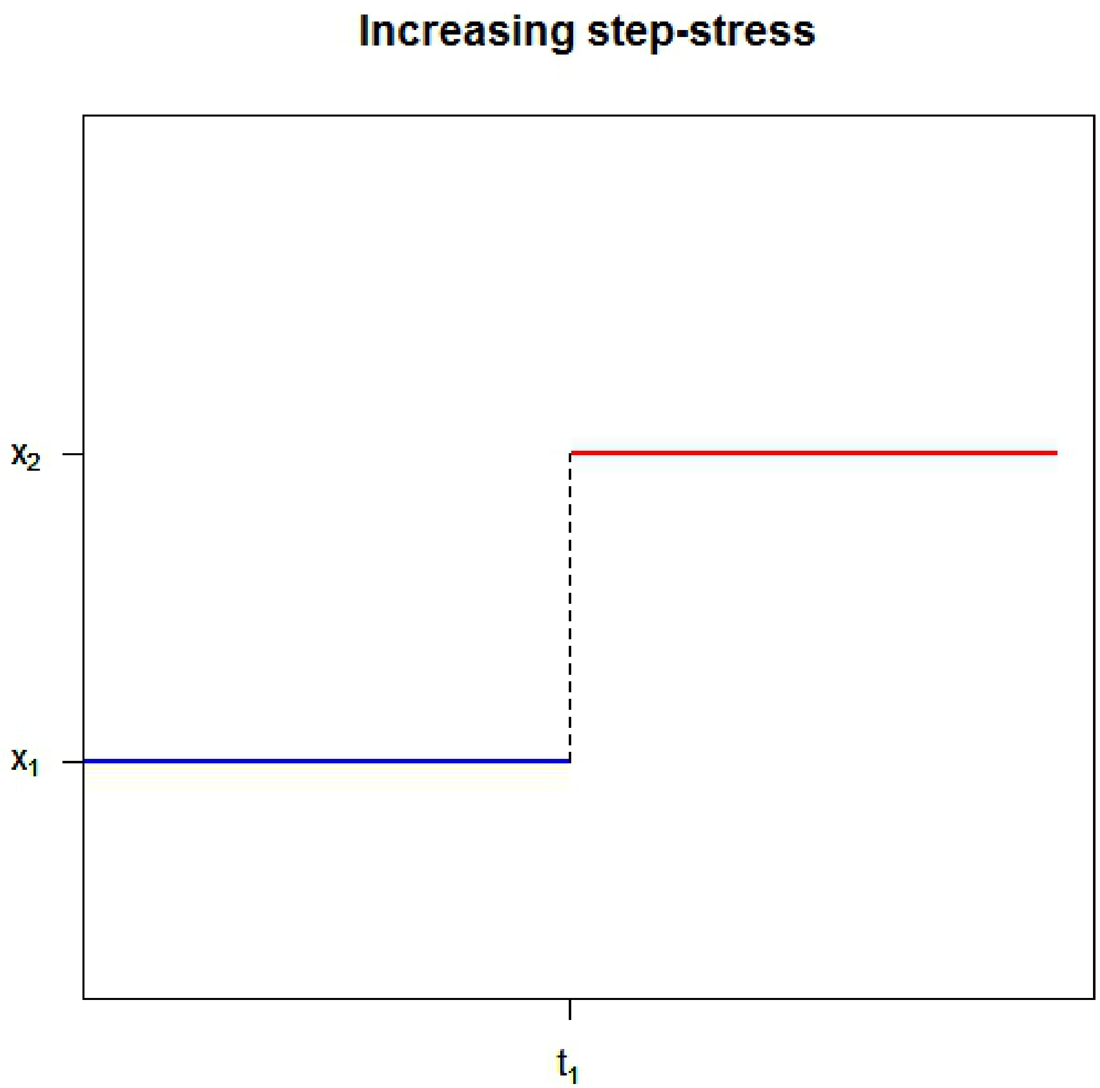

Fig.3. Increasing step-stress for the warm stand-by unit.

Figure 3 shows the **increasing step-stress**. x_1 and x_2 are the stresses corresponding to **warm** and **hot** conditions respectively. Stress x_2 is **accelerated** with respect to stress x_1. The moment t_1 is random in our case.

According to Sedyakin we may consider the model on E_2 for all $s \geq 0$

$$\lambda_{x(\cdot)}(t_1 + s) = \lambda_{x_2}(t_1^* + s). \tag{1}$$

In terms of the survival function $S_{x(\cdot)}(t), x(\cdot) \in E_2$ that satisfies the same **rule of time-shift**

$$S_{x(\cdot)} = \begin{cases} S_{x_1}(t), & 0 \leq t < t_1, \\ S_{x_2}(t - t_1 + t_1^*), & t \geq t_1, \end{cases} \tag{2}$$

where t_1^* is determined by the equation (4).

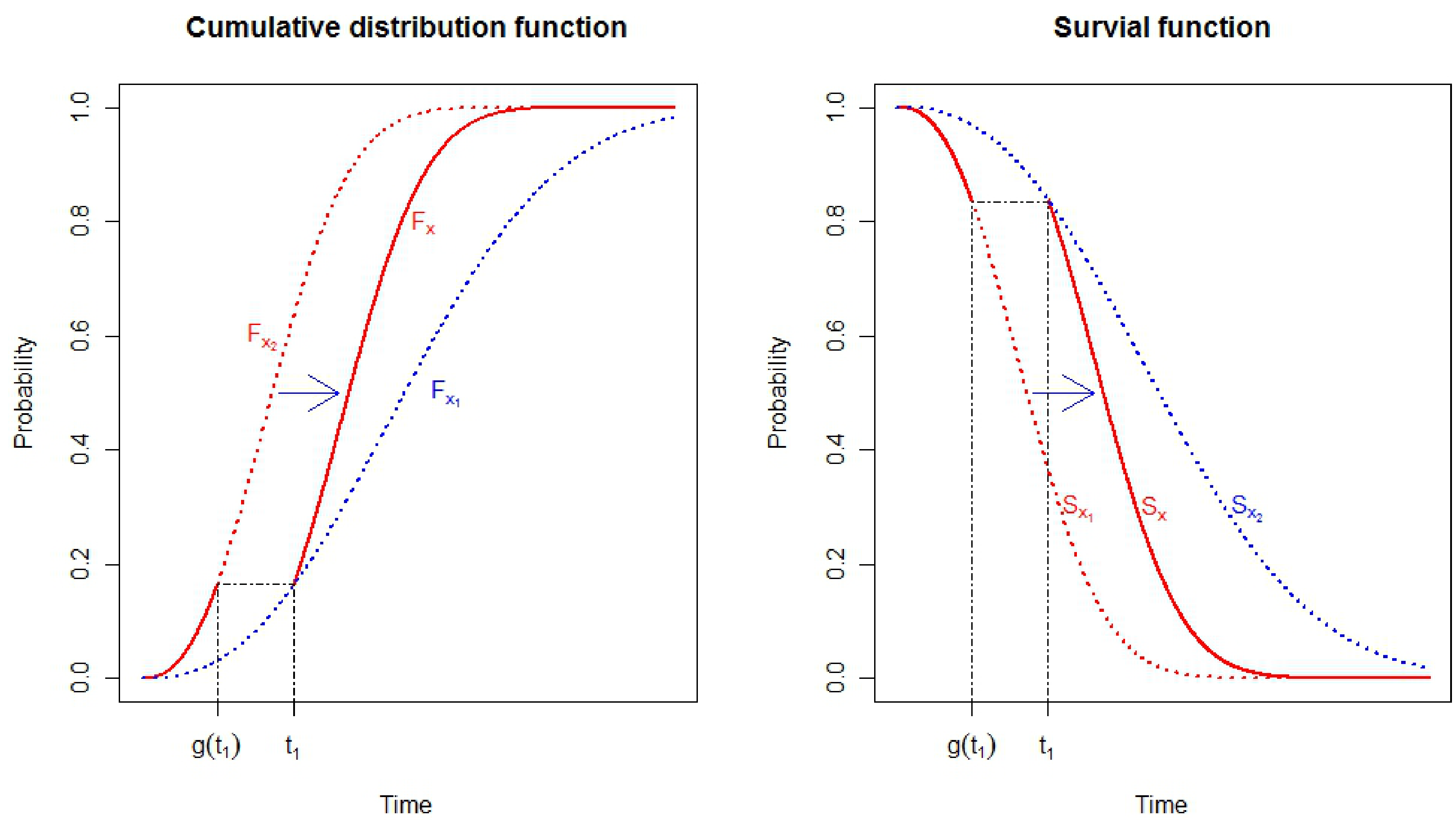

Fig.4. Cumulative distribution function and survival function of the system under Sedyakin's principal.

Denote by T_1, F_1 and f_1 the failure time, the c.d.f. and the probability density function of **the main unit**. The failure times of **the stand-by units** denote by $T_2, \ldots, T_m$. In **"hot"** conditions their c.d.f. are also F_1. In **"warm"** conditions the c.d.f. of T_i is F_2 and the p.d.f. is f_2, $i = 2, \ldots, m$. **If a stand-by unit is switched** to **"hot"** conditions, its c.d.f. **is different** from F_1 and F_2. For $i = 1, 2$ denote by $S_i = 1 - F_i$, $\lambda_i = f_i/S_i$ and $\Lambda_i = -\ln S_i$ the survival function, hazard rate and cumulative hazard, respectively.

The failure time of the system $S(1, m-1)$ is $T^{(m)} = T_1 \vee T_2 \vee \cdots \vee T_m$. Denote by K_j and k_j the c.d.f. and the p.d.f. of $T^{(j)}$, respectively, $(j = 2, \ldots, m)$, $K_1 = F_1$, $k_1 = f_1$. The c.d.f K_j can be written in terms of the c.d.f. K_{j-1} and F_1:

$$K_j(t) = \mathbf{P}(T^{(j)} \leq t) = \int_0^t P(T_j \leq t | T^{(j-1)} = y) dK_{j-1}(y).$$

The **"fluent switch on"** hypothesis H_0 formulated by Bagdonavičius, Masiulaityte and Nikulin (2008) states that

$$f_{T_j|T^{(j-1)}=y}(t) = \begin{cases} f_2(t) & \text{if} \quad t \le y, \\ f_1(t+g(y)-y) & \text{if} \quad t > y; \end{cases} \quad , \quad g(y) = F_1^{-1}(F_2(y)). \tag{8}$$

Conditionally (given $T^{(j-1)} = y$) the hypothesis corresponds to the **Sedyakin's model**. As in the case of the Sedyakin's model $g(y)$ is the moment which in **"hot"** conditions corresponds to the moment y in **"warm"** conditions in the sense that

$$F_1(g(y)) = F_2(y).$$

We note that in the situation considered here **the switch on moments** are **random**.

The model (8) implies that

$$K_j(t) = \int_0^t F_1(t + g(y) - y)dK_{j-1}(y).$$

So the **distribution function** K_m of **the system** with $m-1$ **stand-by units is defined recurrently**.

In particular, if we suppose that the distribution of units functioning in **"warm"** and **"hot" conditions differ only in scale**, i.e. $F_2(t) = F_1(rt)$ for all $t \geq 0$ and some $r > 0$, then $g(y) = ry$. Combining this assumption and the model (8) we have **more strict hypothesis** H_0^*:

$$f_{T_j|T^{(j-1)}=y}(t) = \begin{cases} f_2(t) & \text{if} \quad t \leq y, \\ f_1(t + ry - y) & \text{if} \quad t > y. \end{cases} \tag{9}$$

Conditionally (given $T^{(j-1)} = y$) the hypothesis corresponds to the **AFT** model.

Goodness-of-fit tests for the models (8) and (9), semiparametric and nonparametric estimation procedures are given in Bagdonavičius, Masiulaityte and Nikulin (2008).

2.4. Changing Shape and Scale (CHSS) model

A natural generalization of the **AFT** model (see Nelson (1990), Meeker and Escobar (1998)) is obtained by supposing that **different constant stresses** $x \in E_1$ **influence** not only the **scale** but also the **shape** of the survival distribution: there exist positive functions $\theta(x)$ and $\nu(x)$ on E_1 such that for any $x \in E_1$

$$S(t|x) = S\left\{\left(\frac{t}{\theta(x)}\right)^{\nu(x)} |x_0\right\}; \tag{10}$$

here x_0 is **fixed stress**, for example, **design (usual) stress**. Let us consider generalizations of the model (10) to the case of **time varying stresses**.

We say that the **CHanging Shape and Scale** (**CHSS**) model (see, Bagdonavičius and Nikulin (2002)), holds on E if there exist two positive functions r and ν on E such that for all $x(\cdot) \in E$

$$S(t|x) = S\left(\int_0^t r\{x(\tau)\}\tau^{\nu(x(\tau))-1} d\tau | x_0\right), \quad x \in E.$$

Variation of stress **changes locally** not only the **scale** but also the **shape** of distribution.

In terms of the hazard rate the **CHSS** model can be written in the form:

$$\lambda(t|x) = r\{x(t)\}\, q(\Lambda(t|x))\, t^{\nu(x(t))-1}. \tag{11}$$

This model is not in the class of the **GS** models because the hazard rate $\lambda(t|x)$ depends not only on $x(t)$ and $\Lambda(t|x)$ but also on t. Generally it is recommended to chose $\nu(x) = e^{\gamma^T x}$. Statistical analysis of the **CHSS** model is done in Bagdonavičius, Cheminade, Nikulin (2004), and an interesting application is considererd in Ceci and Mazliak (2004), F.Guérin (2009,2010).

2.5. The Cox or Proportional hazards model

The most popular and most applied regression model in survival analysis is the **proportional hazards** model called also the **Cox model**. The popularity of this model is based on the fact that there **exists simple** semi-parametric estimation procedures which can be used when the form of the survival distribution function is not specified, see Cox(1975). On the other hand the **Cox model** is **rather restrictive** and **is not applicable** when **ratios of hazard rates** under different fixed covariates **are not constant in time**. But in practice **the hazard rates may approach, go away or even intersect**. In such a case **more sophisticated models are needed**. We discuss them in the following sections, see also Huber and Nikulin (2007).

Under the Cox model the hazard rate under a covariate x is given by formula:

$$\lambda(t|x) = e^{\beta^T x(t)} \lambda_0(t), \quad \text{where} \quad x \in E, \tag{12}$$

where $\lambda_0(t)$ is the **baseline** hazard rate function, $\beta = (\beta_1, \cdots, \beta_m)^T$ is the vector of regression parameters. The model implies that hazard rates **under different fixed constant covariates** x_1 and x_2 are **proportional**, i.e. their **the ratio** $R(t, x_1, x_2)$ is **constant over time**:

$$R(t, x_1, x_2) = \exp\{\beta^T (x_1 - x_2)\} = const.$$

Usually **the Cox model** is considered as **semiparametric**: the finite-dimensional parameter β and the baseline hazard function λ_0 are supposed to be **completely unknown**. **Parametric estimation** procedures when λ_0 is taken from some parametric class of functions is scarcely used because the parametric **AFT** model is also simple for analysis and more natural.

The **PH** model **is not much used analyzing failure time regression data in reliability**. The cause is that **the model is not very natural** when the stress is **time-varying**. Indeed, the formula (12) implies that for any t the hazard rate under the time-varying covariate x at the moment t does not depend on the values of the covariate x before the moment t but only on the value of it at this moment, and hence the conditional probability to fail in a time interval $(t, t+s)$ given that a unit is functioning at the moment t depends only on the values of the stress (or covariable) x in that interval but **does not depend on the values of the stress until the moment** t:

$$\mathbf{P}\{T \leq t+s \mid T > t, x(u), 0 \leq u \leq t\} = 1 - e^{-\int_t^{t+s} e^{\beta^T x(u)} \lambda_0(u) du},$$

So it is an **absence of memory model**: it implies that if two groups of units operate under **high** and **low stress**, respectively, for $t \in [0, t_1]$, and **both operate under the same stress after the moment** t_1, then both groups have **the same hazard rate** for any $t > t_1$. It means that **aging** is the same under high and low stress conditions.

Nevertheless, in survival analysis the **PH** model **usually works quite well**, **because** the values of covariates under which estimation of survival is needed are **in the range** of covariate values **used in experiments**.

So the use of a not very exact but simple model often is preferable to the use of more adequate but complicated model. It is similar with application of **linear regression models** in classical regression analysis: the mean of dependent variable is rarely a linear function of independent variables but the linear approximation works reasonably well **in some range** of independent variable values. It was already noted that in reliability, **accelerated life testing** in particular, the choice of a good model is much more important than in survival analysis, see Huber and Nikulin (2007).

The paper of **Sir David Cox** (1975) which gives **semiparametric estimation procedure** of the regression parameters β maximizing partial likelihood function, stimulated enormous quantity of developments in the theory of semiparametric estimation and is the most cited paper in the statistical literature.

In the following sections we consider extensions of the Cox model.

2.6. Generalized Proportional Hazards (GPH) models

Under the **GPH** (see, Bagdonavičius and Nikulin (1999), (2002)) model the hazard rate under a covariate realization x in given by formula

$$\lambda(t|x) = r\,(x(t))\,q\,(\Lambda(t|x))\,\lambda_0(t), \quad x \in E, \tag{13}$$

where q is positive, with $q(0) = 1$. The value $\Lambda(t|x) = -\ln S(t|x)$ characterizes influence of the stress x on survival until the moment t and may be called the **resource** used until this moment. The **GPH** model means that the the hazard rate (or, equivalently, resource usage rate) at the moment t is proportional not only to a function of the stress applied at this moment but also on a function of the resource used until t and to a baseline hazard which does not depend on stress.

The model allows the ratios of the hazard rates under constant stresses to be increasing or decreasing. Note that it contains the **PH** model ($Q(u) \equiv 1$) and the **AFT** model ($\lambda_0(t) \equiv 1$) as particular cases.

The possible parametrizations of the function q are the following:

$$q(u) = (1+u)^{1-\gamma}, \quad q(u) = e^{\gamma u}, \quad q(u) = (1+\gamma u)^{-1}.$$

Under these parametrizations the situations with approaching and going away hazard rates under various constant covariates may be modelled.

In the case of the first and the third models the hazard rate is proportional to the power function of the resource, in the case of the second model - to the exponential function of the resource.

Take notice that in the case of all three models generalizing the**PH** model only one complementary one-dimensional parameter γ is included.

Statistical analysis of **GPH** models are given in Bagdonavičius and Nikulin (1999), where modified partial likelihood function was introduced. Dabrowska (2006) studied properties of **MPL** estimators for the model $\Lambda(t|z) = A(\Lambda(t), \theta|z)$, supposing that A is a known function. The model (13) belongs to a class of such models.

2.7. Models with cross-effects of survival functions

Hazard rates under different values of covariates may cross. If the hazard rates of two populations **do not cross** then we can state that **the risk of failure of one population is smaller** then that of the second in time interval $[0, \infty)$. So one of populations is **"uniformly more reliable"**.

Hsieh (2001) considered the following model:

$$\lambda(t|x) = e^{\beta^x(t)+\gamma^T \tilde{x}(t)} \{\Lambda(t)\}^{e^{\gamma^T \tilde{x}(t)}-1} \lambda(t), \quad \Lambda(t) = \int_0^t \lambda(u)du,$$

λ is unknown baseline hazard rate, $\tilde{x} = (x_{i_1}, \ldots, x_{i_k})^T$, $1 \leq i_1 < \cdots < i_k \leq s$.

The model **does not contain interesting alternatives to crossing**: the hazard rates coincide if $\beta = \gamma = 0$, are proportional if $\gamma = 0$. In all other cases the hazards rates under different constant covariates cross (**no approaching, no going away, no converging**). Bagdonavičius and Nikulin (2002, 2006), Bagdonavičius, Hafdi and Nikulin (2004) considered more versatile simple cross effect (**SCE**) model:

$$\lambda(t|x) = e^{\beta^T x(t)} \{1 + \Lambda(t|x)\}^{1 - e^{\gamma^T x(t)}} \lambda(t), \quad \Lambda(t|x) = \int_0^t \lambda(u|x) du.$$

he hazard rates **may be proportional, meet at zero, cross or go away** but can not converge and meet at infinity.

Zeng and Lin (2007) include an additional parameter to the above mentioned models: in terms of cumulative hazards their models are written ($\tilde{x}$ is supposed to be constant in time) respectively

$$\Lambda(t|x) = G\left(\left(\int_0^t e^{\beta^T x(u)} d\Lambda(u)\right)^{e^{\gamma^T \tilde{x}}}\right) \tag{14}$$

and

$$\Lambda(t|x) = G\left(\left(1 + \int_0^t e^{\beta^T x(u)} d\Lambda(u)\right)^{e^{\gamma^T \tilde{x}}}\right) - G(1). \tag{15}$$

Here

$$G(u) = \frac{(1+u)^{\rho} - 1}{\rho}, \quad \rho > 0, \quad G(x) = \log(1+u), \quad \rho = 0,$$

(**Box-Cox transformation**) or

$$G(x) = \frac{\log(1+ru)}{r}, \quad r > 0, \quad G(u) = u, \quad r = 0.$$

Taking $G(u) = u$ the models of Hsieh and Bagdonavičius&Nikulin are obtained from (14) and (15) respectively.

Henderson (2007) remarks that it **is difficult** to see the role of three parameters in these models. Estimation procedure in such general models is also very complicated.

In the recent paper of Bagdonavičius, Levuliene and Nikulin (2008) a **flexible and simple semiparametric model** including not only possibility of hazard rates crossing but also most likely alternatives, stating that hazard rates go away, are proportional, approach or converge, is proposed:

$$\lambda(t|x) = \frac{e^{\beta^T x(t) + \Lambda(t) e^{\gamma^T \tilde{x}(t)}}}{1 + e^{\beta^T x(t) + \gamma^T \tilde{x}(t)} [e^{\Lambda(t) e^{\gamma^T \tilde{x}(t)}} - 1]} \lambda(t),$$

$$\Lambda(t) = \int_0^t \lambda(u) du;$$

λ is unknown baseline hazard rate; $\tilde{x} = (x_{i_1}, \ldots, x_{i_k})^T$, $1 \leq i_1 < \cdots < i_k \leq s$, is a subset of $x = (x_1, \ldots, x_s)^T$.

This model does not contain additional parameters ρ or r given Zeng and Lin (2007).

Bagdonavičius, Levuliene and Nikulin (2008) give estimation procedures and tests for this model. Many lifetime regression models can be found in Meeker and Escobar (1997), Bagdonavičius and Nikulin (2002), Lawless (2003), Duchesne and Rosenthal (2003), Nelson (2004), Martinussen and Scheike (2006), Scheike (2006), Huber-Carol and Nikulin (2008).

3. Failure time-degradation models with explanatory variables

Suppose that the following data may be available for reliability characteristics estimation: failure times (possibly censored), explanatory variables (covariates, stresses) and the values of some observable quantity characterizing the degradation of units.

We call **a failure non-traumatic** when the degradation attains a **critical level** z_0. Other failures are called **traumatic**. Denote $T^{(0)}$ the **moment of non-traumatic failure**.

The failure rate of traumatic failure may depend on **covariates, degradation level and time**. Good reviews on failure time-degradation data modeling are given in Singpurwalla (1995), Lehmann (2004), Yashin (2004), Nikulin, Limnios, Balakrishnan, Kahle, Huber (2010). We develop the models given in these excellent papers.

Suppose that **under fixed constant covariate** the degradation is non-decreasing stochastic process $Z(t), t \geq 0$ with **right continuous trajectories and with finite left hand limits** (cadlag). For example, the degradation process $Z(t)$ can be modelled by the **linear path model** $Z(t) = t/A$, where A is a positive random variable. Denote by $T^{(tr)}$ the **moment of the traumatic failure** and by

$$\lambda^{(tr)}(t|Z) = \lambda^{(tr)}(t|Z(s), 0 \leq s \leq t)$$

the **conditional hazard rate of the traumatic failures** given degradation.

Suppose that this conditional hazard rate has **two additive components**: **one related to observed degradation values**, other – to **non-observable degradation (aging) and to possible shocks causing sudden traumatic failures**. For example, **observable degradation** of tires is the wear of the protector.

The failure rate of tire explosion depends on thickness of the protector, on non-measured degradation level of other tire components and on intensity of possible shocks (hitting a kerb, nail). So **the hazard rate of the traumatic failure** is modelled as follows:

$$\lambda^{(tr)}(t|Z) = \lambda(Z(t)) + \mu(t).$$

The function $\lambda(z)$ **characterizes the dependence of the rate of traumatic failures on degradation**.

Suppose that **covariates influence degradation rate and traumatic event intensity**. In such a case **the model needs to be modified**.

Let $x = (x_1, \ldots, x_m)^T$ be a vector of s possibly time dependent one-dimensional covariates. We assume in what follows that x_i are deterministic or realizations of bounded right continuous with finite left hand limits stochastic processes.

Denote informally by $Z(t|x)$ **the degradation level** at the moment t for units functioning under the covariate x. We suppose that the **covariates influence locally** the scales of the traumatic failure time distribution component related to non-observable degradation (aging) and to possible shocks, i.e. the **AFT** model is true for this component.

Let us explain it in detail. Denote by

$$S_1^{(tr)}(t|Z) = \exp\{-\int_0^t \lambda[Z(u)]du\}, \quad S_2^{(tr)}(t) = \exp\{-\int_0^t \mu(u)du\}$$

the survival functions corresponding to the hazard rates $\lambda(Z(u))$ and $\mu(u)$. The first survival function is conditional given the degradation.

The **AFT** model defines the following relation of the second survival function and the covariates:

$$S_2^{(tr)}(t|x) = S_2(\int_0^t e^{\gamma^T x(s)}\, ds);$$

the parameters γ have the same dimension as x.

Set

$$f(t|x) = f(t, x, \beta) = \int_0^t e^{\beta^T x(u)}\, du,$$

and denote by $g(t, x, \beta)$ the inverse of $f(t, x, \beta)$ with respect to the **first argument**. If $x = const$ then

$$f(t, x, \beta) = e^{\beta^T x} t, \quad g(t, x, \beta) = e^{-\beta^T x} t$$

The function $f(t, x, \beta)$ is **time transformation** in dependence on x. For two units functioning under different covariates x and y two moments t_1 and t_2, respectively, **are equivalent in the sense of the probability** to have the failure if they veryfy the equality

$$f(t_1, x, \beta) = f(t_2, y, \beta)$$

We are able to consider the following model for **degradation**

process under covariates:

$$Z(t|x) = Z(f(t, x, \beta)).$$

The covariates have **double influence** on the distribution of the first traumatic failure component: **via degradation and directly.**

So we combine the **AFT** and the **proportional hazards models**:

$$S_1^{(tr)}(t|x,Z) = \exp\{-\int_0^t e^{\tilde{\beta}^T x(u)} \lambda(Z(u|x)du\}.$$

Denote by

$$S^{(tr)}(t|x,Z) = \mathbf{P}(T^{(tr)} > t|x(u), Z(u|x), 0 \le u \le t),$$

$$\lambda^{(tr)}(t|x,Z) = -\frac{d}{dt} \ln S^{(tr)}(t|x,Z)$$

the **conditional distribution function** and **the failure rate of the traumatic failure given the covariates and the degradation**.

So we consider the following **failure time-degradation-covariate model**:

$$\lambda^{(tr)}(t|x, Z) = e^{\tilde{\beta}^T x(t)} \lambda(Z(f(t, x, \beta))) + e^{\gamma^T x(t)} \mu(f(t, x, \gamma)).$$

Under this model

$$S^{(tr)}(t|x, Z) = \mathbf{P}(T^{(tr)} > t|x(u), Z(u|x), 0 \le u \le t) =$$

$$\exp\left\{-\int_0^t e^{\tilde{\beta}^T x(u)} \lambda(Z(u|x))\, du - H(f(t, x, \gamma))\right\},$$

$$H(t) = \int_0^t \mu(u) du.$$

Denote

$$T^{(0)} = \inf\{t : Z(t|x) \geq z_0\}$$

the moment of **non-traumatic failure**, and let

$$S^{(0)}(t|x) = \mathbf{P}\{T^{(0)} > t|x(u), 0 \leq u \leq t\} = \mathbf{P}\{Z(t|x) < z_0|x(u), 0 \leq u \leq t\}$$

be the **survival function of the random variable** $T^{(0)}$ under the covariate x. Let

$$\mathbf{T} = \min(\mathbf{T^{(0)}}, \mathbf{T^{(tr)}})$$

be the **time of the unit failure**. It may be **traumatic** or **non-traumatique**. The **survival function of the failure time** T under the covariate x is

$$S(t|x) = \mathbf{P}(T > t|x) = \mathbf{E}S(t|x, Z),$$

$$S(t|x, Z) = \mathbf{1}_{\{Z(t|x) < z_0\}} S^{(tr)}(t \mid x, Z).$$

In the partial case $z_0 = \infty$ the survival functions $S(t|x, Z)$ and $S^{(tr)}(t|x, Z)$ coincide. **Mean failure time under the covariate** x is

$$e(x) = \mathbf{E}(T|x) = \mathbf{E}(\mathbf{E}(T|x, Z)),$$

$$\mathbf{E}(T|x, Z) = \int_0^{g(h(z_0),x,\beta)} S(t \mid x, Z)dt.$$

The probability of non-traumatic failure under the covariate x in the interval $[0, t]$:

$$P^{(0)}(t|x) = \mathbf{E}P^{(0)}(t|x, Z),$$

$$P^{(0)}(t|x, Z) = \mathbf{1}_{\{Z(t|x) \geq z_0\}} S(g(h(z_0), x, \beta) \mid x, Z).$$

In particular, the **probability of non-traumatic failure** under the covariate x in the interval $[0,\infty)$ is obtained.

The **probability of traumatic failure** under the covariate x in the interval $[0,t]$ is

$$P^{(tr)}(t|x) = \mathbf{E}P^{(tr)}(t|x,Z),$$

$$P^{(tr)}(t|x,Z) = 1 - S(t \wedge g(h(z_0),x,\beta) \mid x,Z).$$

Analyzing of failure time-degradation data **requires** not only relate the probability of failures with degradation and covariates **but also models for degradation process**.

The most applied stochastic processes describing degradation are general path models and time scaled **stochastic processes with stationary and independent increments** such as the **gamma process, shock processes and Wiener process with a drift**:

General path models

: $Z(t) = g(t, A, \theta)$, where g is a deterministic function and $A = (A_1, \ldots, A_s)$ is a finite dimensional random vector and θ is a finite dimensional non-random parameter. The form of the function g may be suggested by the form of **individual degradation curves**. The **degradation under the covariate** x is modelled by

$$Z(t|x) = g(f(t, x, \beta), A),$$

$$m(t|x) = \mathbf{E}g(f(t, x, \beta), A).$$

Methods of estimation from degradation data are given in Carey and Koenig (1991), Lu and Meeker (1993), Meeker and Escobar (1998). Bagdonavičius, Bikelis and Kazakevičius (2002)), Bagdonavičius *et al* (2007, 2010) considered estimation from failure time-degradation data.

Time scaled gamma process

: $Z(t) = \sigma^2 \gamma(t)$, where $\gamma(t)$ is a process with independent increments such that for any fixed $t > 0$

$$\gamma(t) \sim G\big(1, \nu(t)\big), \quad \nu(t) = \frac{m(t)}{\sigma^2},$$

i.e. $\gamma(t)$ has the **gamma distribution** with the density

$$p_{\gamma(t)}(x) = \frac{x^{\nu(t)-1}}{\Gamma\big(\nu(t)\big)}\, e^{-x}, \quad x \geq 0,$$

where $m(t)$ is an increasing function.

Then

$$Z(t|x) = \sigma^2\,\gamma(f(t,x,\beta)).$$

The **mean degradation** and the **covariances** under the covariate x are

$$m(t|x) = \mathbf{E}(Z(t|x)) = m\left(f(t,x,\beta)\right),$$

and

$$\mathbf{Cov}(Z(s|x), Z(t|x)) = \sigma^2 m(f(s \wedge t, x, \beta)).$$

Bagdonavičius and Nikulin (2002b) considered estimation from failure time-degradation data, Lawless and Crowder (2004), Padgett and Tomlinson (2005) considered estimation from degradation data.

Time scaled Wiener process with a drift

: $Z(t) = m(t) + \sigma W(m(t))$, where W denotes the standard Wiener motion, i.e. a process with independent increments such that $W(t) \sim N(0, t)$. Then

$$Z(t|x) = m(f(t, x, \beta)) + \sigma W(m(f(t, x, \beta))).$$

The mean degradation and the covariances under the covariate x are

$$m(t|x) = m\left(f(t, x, \beta)\right), \quad \mathbf{Cov}(Z(s|x), Z(t|x)) = \sigma^2 m(f(s \wedge t, x, \beta)).$$

Doksum and Normand (1995), Lehmann,(2001), (2005), Padgett and Tomlinson (2004)) considered estimation from degradation data.

Shock processes

. Assume that degradation results from shocks, each of them leading to an increment of degradation. Let $T_n, (n \geq 1)$ be the time of the nth shock and X_n the nth increment of the degradation level. Denote by $N(t)$ the number of shocks in the interval $[0, t]$. Set $X_0 = 0$. The degradation process is given by

$$Z(t) = \sum_{n=1}^{\infty} \mathbf{1}\{T_n \leq t\} X_n = \sum_{n=0}^{N(t)} X_n.$$

Kahle and Wendt (2006) model T_n as the moments of transition of the **doubly stochastic Poisson process**, i.e. they suppose that the distribution of the number of shocks up to time t is given by

$$P\{N(t) = k\} = \mathbf{E}\left\{ \frac{(Y\eta(t))^k}{k!} \exp\{-Y\eta(t)\} \right\},$$

where $\eta(t)$ is a deterministic function and Y is a nonnegative random variable with finite expectation. If Y is non-random, N is **non-homogenous Poisson process**, in particular, when $\eta(t) = \lambda t$, N is **homogenous Poisson process**.

If $\eta(t) = t$, then N is a **mixed Poisson process**. Other models for η may be used, for example, $\eta(t) = t^{\alpha}$, $\alpha > 0$. The random variable Y is taken from some parametric class of distributions.

Kahle and Lehmann (1998), Wendt and Kahle (2004), Kahle and Wendt (2006) considered parametric estimation from degradation data. Lehmann (2005, 2006) considered estimation from failure time-degradation data.

Harlamov (2004) discusses inverse gamma-process as a wear model. Zacks (2004) gives failure distributions associated with general compound renewal damage processes. It is interesting to see also the recent book

"Advances in Degradation Modeling. Applications to Reliability, Survival Analysis and Finace" (Eds. Nikulin, Limnios, Balakrishnan, Kahle, Huber), Birkhauser, (2010).

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